

# BICHA- RACTERISTICS AND FOURIER INTEGRAL OPERATORS IN KASNER SPACETIME

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The scalar wave equation in Kasner spacetime is solved, first for a particular choice of Kasner parameters, by relating the integrand in the wave packet to the Bessel functions. An alternative integral representation is also displayed, which relies upon the method of integration in the complex domain for the solution of hyperbolic equations with variable coefficients. In order to study the propagation of wave fronts, we integrate the equations of bicharacteristics which are null geodesics, and we are able to express them, for the first time in the literature, with the help of elliptic integrals for another choice of Kasner parameters. For generic values of the three Kasner parameters, the solution of the Cauchy problem is built through a pair of integral operators, where the amplitude and phase functions in the integrand solve a coupled system of partial differential equations. The first is the so-called transport equation, whereas the second is a nonlinear equation that reduces to the eikonal equation if the amplitude is a slowly varying function. Remarkably, the analysis of such a coupled system is proved to be equivalent to building first an auxiliary covariant vector having vanishing divergence, while all nonlinearities are mapped into solving a covariant generalization of the Ermakov-Pinney equation for the amplitude function. Last, from a linear set of equations for the gradient of the phase one recovers the phase itself. This is the parametrix construction that relies upon Fourier-Maslov integral operators, but with a novel perspective on the nonlinearities in the dispersion relation. Furthermore, the Adomian method for nonlinear partial differential equations is applied to generate a recursive scheme for the evaluation of the amplitude function in the parametrix. The resulting formulas can be used to build self-dual solutions to the field equations of noncommutative gravity, as has been shown in

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the recent literature.

*Keywords:* Cauchy problem, bicharacteristics, parametrix

## 1. Introduction

Although it is well known that the wavelike phenomena of classical physics are ruled by hyperbolic equations, there are at least two modern motivations for studying the scalar wave equation on curved Lorentzian four-manifolds. They are as follows.

(i) In the course of studying the Einstein vacuum equations in four dimensions, i.e.

$$R_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}R = 0 \implies R_{\alpha\beta} = 0, \quad (1.1)$$

it was conjectured in Ref. [1] that they admit local Cauchy developments for initial data<sup>a</sup> sets  $(\Sigma_0, h, k)$  with locally finite  $L^2$  curvature and locally finite  $L^2$  norm of the first covariant derivatives of  $k$ . This means that the spacetime constructed by evolution from smooth data can be smoothly continued, together with a time foliation, as long as the curvature of the foliation and the first covariant derivatives of its extrinsic curvature remain  $L^2$ -bounded on the leaves of the foliation [2]. The proof that this is indeed the case relies on a number of technical ingredients, including the construction of a parametrix (an approximate Green function of the wave operator, that provides a progressive wave representation [3] for solutions of the wave equation) for solutions of the homogeneous wave equation

$$\square_g \phi = g^{\mu\nu} \nabla_\mu \nabla_\nu \phi = 0 \quad (1.2)$$

on a fixed Einstein vacuum background. One has then to obtain control of the parametrix and of its error term by using only the fact that the curvature tensor is bounded in  $L^2$  [4]. Note that, at a deeper level, the metric  $g$  can be viewed to determine the elliptic or hyperbolic nature<sup>b</sup> of the operator  $g^{\mu\nu} \nabla_\mu \nabla_\nu$ , where  $\nabla_\mu$

<sup>a</sup>The Cauchy problem consists in finding a Lorentzian four-metric  $g$  satisfying Eq. (1.1) such that the metric  $h$  induced by  $g$  on a given spacelike hypersurface  $\Sigma_0$  and the extrinsic-curvature tensor  $k$  of  $\Sigma_0$  are prescribed.

<sup>b</sup>If  $M$  is a connected, four-dimensional, Hausdorff four-manifold of class  $C^\infty$ , a linear partial differential operator is a linear map

$$L : u \in C^\infty(M) \rightarrow (Lu) \in C^k(M),$$

with coefficients  $a^{i_1 \dots i_m}$  given by functions of class  $C^k$ . The *characteristic polynomial* of the operator  $L$  at a point  $x \in M$  is

$$H(x, \xi) = \sum a^{i_1 \dots i_m}(x) \xi_{i_1} \dots \xi_{i_m},$$

where  $\xi_i$  is a cotangent vector at  $x$ . The cone in the cotangent plane  $T_x^*$  at  $x$  defined by

$$H(x, \xi) = 0$$

is called the characteristic cone (or conoid). By construction, such a cone is independent of the choice of coordinates, because the higher order terms (also called leading or principal symbol) of  $L$  transform into higher-order terms by a change of coordinates. The operator  $L$  is said to

can denote covariant differentiation with respect to the Levi-Civita connection on spacetime, or on a vector bundle over spacetime, depending on our needs. When  $g$  is Riemannian, i.e. positive-definite, this operator is minus the Laplacian, whereas if  $g$  is Lorentzian, one gets the wave operator. Note also that, in four-dimensional manifolds, our Lorentzian world lies in between two other options, i.e. a Riemannian metric  $g$  with signature 4 and elliptic operator  $g^{\mu\nu}\nabla_\mu\nabla_\nu$ , and a ultrahyperbolic metric  $g$  with signature 0 and ultrahyperbolic operator  $g^{\mu\nu}\nabla_\mu\nabla_\nu$ . In the so-called Euclidean (or Riemannian) framework used by quantum field theorists in functional integration, where the metric is positive-definite, the most fundamental differential operator is however the Dirac operator, obtained by composition of Clifford multiplication with covariant differentiation. Its leading symbol is therefore Clifford multiplication, and it generates all elliptic symbols on compact Riemannian manifolds [5]. This reflects the better known property according to which, out of the Dirac operator and its (formal) adjoint, one can define two operators of Laplace type, as well as powers of these operators.

(ii) Recent work on the self-dual road to noncommutative gravity with twist has found it useful to start from a classical, undeformed spacetime which is a self-dual solution of the vacuum Einstein equation, e.g. a Kasner spacetime [6]. Within that framework, it is of interest to solve first the scalar wave equation in such a Kasner background. Since such a task was only outlined in Ref. [6], we find it appropriate to develop a systematic calculus in the present paper.

Relying in part upon Ref. [6], we begin by considering the scalar wave equation (1.2) for a classical scalar field  $\phi$  when the Kasner<sup>c</sup> parameters  $p_1, p_2, p_3$  take the values 1, 0, 0, respectively, i.e.

$$\left(-\frac{\partial^2}{\partial t^2} - \frac{1}{t}\frac{\partial}{\partial t} + \frac{1}{t^2}\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\phi = 0, \quad (1.3)$$

be hyperbolic at  $x$  if there exists a domain  $\Gamma_x$ , a convex open cone in  $T_x^*$ , such that every line through  $\lambda \in \Gamma_x$  cuts the characteristic cone in  $m$  real distinct points. In particular, second-order differential operators with higher-order terms

$$g^{\alpha\beta}(x)\frac{\partial}{\partial x^\alpha}\frac{\partial}{\partial x^\beta}$$

are hyperbolic at  $x$  if and only if the cone defined by

$$H_2(x, \xi) \equiv g^{\alpha\beta}(x)\xi_\alpha\xi_\beta = 0$$

is convex, i.e., if the quadratic form  $H_2(x, \xi)$  has signature  $(1, n-1)$ .

<sup>c</sup>With this particular choice of parameters, we are working on an edge of Minkowski spacetime, i.e., Rindler space. The literature on quantum field theory and accelerated observers has considered in detail such a space, but in our paper the emphasis is on partial differential equations in classical physics, hence we do not strictly need ideas from quantum physics. It would have been helpful to derive the world function in Kasner spacetime (Appendices B and C) from the world function in Minkowski spacetime, and similarly for the Green function of Hadamard type, but we have been unable to achieve this. Hence we have limited ourselves to strict use of Kasner geometry.

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where  $\phi$  admits the integral representation

$$\phi(t, x, y, z) = \int_{-\infty}^{\infty} d\xi_1 \int_{-\infty}^{\infty} d\xi_2 \int_{-\infty}^{\infty} d\xi_3 A(\xi_1, \xi_2, \xi_3, t) e^{i(\xi_1 x + \xi_2 y + \xi_3 z)}. \quad (1.4)$$

One can then set [6]

$$A(\xi_1, \xi_2, \xi_3, t) = \frac{1}{\sqrt{t}} W(\xi_1, \xi_2, \xi_3, t), \quad (1.5)$$

where  $W(\xi_1, \xi_2, \xi_3, t)$  has to solve, for consistency, the equation [6]

$$\left[ \frac{\partial^2}{\partial t^2} + \frac{(\frac{1}{4} + (\xi_1)^2)}{t^2} + (\xi_2)^2 + (\xi_3)^2 \right] W(\xi_1, \xi_2, \xi_3, t) = 0, \quad (1.6)$$

and the  $\frac{1}{\sqrt{t}}$  term in the factorization (1.5) ensures that, in Eq. (1.6), the first derivative of  $W$  is weighed by a vanishing coefficient. This is a sort of canonical form of linear second-order ordinary differential equations with variable coefficients (see Section 10.2 of Ref. [7]), and Eq. (1.6) can be viewed as a 3-parameter family of such equations, the parameters being the triplet  $\xi_1, \xi_2, \xi_3$ .

Section II relates Eq. (1.6) to the Bessel functions, studies a specific choice of Cauchy data and eventually solves Eq. (1.3) through an integral representation that relies upon integration in the complex domain. Section III evaluates the bicharacteristics in Kasner spacetime, relating them to elliptic integrals, while Sec. IV builds the parametrix of our scalar wave equation through a pair of integral operators where the integrand consists of amplitude and phase functions. Concluding remarks and open problems are presented in Sec. V, while relevant background material is described in the Appendices.

## 2. Solving the wave equation with Kasner parameters (1, 0, 0)

### 2.1. Relation with Bessel functions

From now on, we therefore study until the end of next subsection the ordinary differential equation

$$\left[ \frac{d^2}{dt^2} + \frac{(\frac{1}{4} + (\xi_1)^2)}{t^2} + (\xi_2)^2 + (\xi_3)^2 \right] W(t) = 0. \quad (2.1)$$

This is a particular case of the differential equation

$$\left[ \frac{d^2}{dt^2} + \frac{(1 - 2\alpha)}{t} \frac{d}{dt} + \beta^2 + \frac{(\alpha^2 - \nu^2)}{t^2} \right] f(t) = 0, \quad (2.2)$$

which is solved by the linear combination

$$f(t) = C_1 t^\alpha J_\nu(\beta t) + C_2 t^\alpha Y_\nu(\beta t). \quad (2.3)$$

By comparison of Eqs. (2.1) and (2.2) we find

$$\alpha = \frac{1}{2}, \quad \beta = \sqrt{(\xi_2)^2 + (\xi_3)^2}, \quad \nu = i\xi_1, \quad (2.4)$$

and hence, in light of what we pointed out at the end of Sec. I, our partial differential equation (1.4) is solved by replacing  $C_1$  and  $C_2$  in (2.3) by some functions  $Z_1(\xi_1, \xi_2, \xi_3)$  and  $Z_2(\xi_1, \xi_2, \xi_3)$ , whose form depends on the choice of Cauchy data, i.e. (see Sec. III)

$$W(\xi_1, \xi_2, \xi_3, t) = Z_1(\xi_1, \xi_2, \xi_3) \sqrt{t} J_{i\xi_1}(t \sqrt{(\xi_2)^2 + (\xi_3)^2}) + Z_2(\xi_1, \xi_2, \xi_3) \sqrt{t} Y_{i\xi_1}(t \sqrt{(\xi_2)^2 + (\xi_3)^2}). \quad (2.5)$$

The Bessel function  $Y_{i\xi_1}$  is not regular at  $t = 0$  and hence, by using this representation, we are considering an initial time  $t_0 > 0$ . We use the linearly independent Bessel functions  $J_{i\xi_1}$  and  $Y_{i\xi_1}$  which describe accurately the time dependence of the integrand in Eq. (1.4). Note that the three choices

$$(p_1 = 1, p_2 = p_3 = 0), (p_2 = 1, p_1 = p_3 = 0), (p_3 = 1, p_1 = p_2 = 0)$$

are equivalent, since the three coordinates  $x, y, z$  in the scalar wave equation [6] are on equal footing. Only the calculational details change. More precisely, on choosing  $p_3 = 1, p_1 = p_2 = 0$ , one finds

$$\beta = \sqrt{(\xi_1)^2 + (\xi_2)^2}, \quad \nu = i\xi_3,$$

whereas, upon choosing  $p_2 = 1, p_1 = p_3 = 0$ , one finds

$$\beta = \sqrt{(\xi_1)^2 + (\xi_3)^2}, \quad \nu = i\xi_2.$$

## 2.2. Role of Cauchy data

The task of solving our wave equation (1.3) can be accomplished provided that one knows the Cauchy data

$$\Phi_0 \equiv \phi(t_0, x, y, z), \quad \Phi_1 \equiv \frac{\partial \phi}{\partial t}(t = t_0, x, y, z). \quad (2.6)$$

Indeed, from our Eqs. (1.4), (1.5) and (2.5), one finds (denoting by an overdot the partial derivative with respect to  $t$ )

$$\begin{aligned} A(\xi_1, \xi_2, \xi_3, t_0) &= (2\pi)^{-3} \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} dz \Phi_0 e^{-i(\xi_1 x + \xi_2 y + \xi_3 z)} \\ &= Z_1(\xi_1, \xi_2, \xi_3) J_{i\xi_1}(t_0 \sqrt{(\xi_2)^2 + (\xi_3)^2}) + Z_2(\xi_1, \xi_2, \xi_3) Y_{i\xi_1}(t_0 \sqrt{(\xi_2)^2 + (\xi_3)^2}) \quad (2.7) \\ \dot{A}(\xi_1, \xi_2, \xi_3, t_0) &= (2\pi)^{-3} \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} dz \Phi_1 e^{-i(\xi_1 x + \xi_2 y + \xi_3 z)} \\ &= \sqrt{(\xi_2)^2 + (\xi_3)^2} \left[ Z_1(\xi_1, \xi_2, \xi_3) \dot{J}_{i\xi_1}(t_0 \sqrt{(\xi_2)^2 + (\xi_3)^2}) \right. \\ &\quad \left. + Z_2(\xi_1, \xi_2, \xi_3) \dot{Y}_{i\xi_1}(t_0 \sqrt{(\xi_2)^2 + (\xi_3)^2}) \right]. \quad (2.8) \end{aligned}$$

Equations (2.7) and (2.8) are a linear system of algebraic equations to be solved for  $Z_1$  and  $Z_2$ , and they can be studied for various choices of Cauchy data. For

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example, inspired by the simpler case of scalar wave equation in two-dimensional Minkowski spacetime, we may consider the Cauchy data [8]

$$\Phi_0 \equiv e^{-\frac{(x^2+y^2+z^2)}{2L^2}} (\cos \gamma_1 x) (\cos \gamma_2 y) (\cos \gamma_3 z), \quad (2.9)$$

$$\Phi_1 \equiv 0, \quad (2.10)$$

where  $L$  has dimension of length. Thus, by virtue of the identity

$$\int_{-\infty}^{\infty} dx e^{-i\xi x} e^{-\frac{x^2}{2L^2}} (\cos \xi_0 x) = \sqrt{2\pi} \frac{L}{2} \left[ e^{-\frac{L^2}{2}(\xi - \xi_0)^2} + e^{-\frac{L^2}{2}(\xi + \xi_0)^2} \right], \quad (2.11)$$

we obtain from (2.7) and (2.9)

$$A(\xi_1, \xi_2, \xi_3, t_0) = (2\pi)^{-\frac{3}{2}} \left( \frac{L}{2} \right)^3 \prod_{i=1}^3 \left[ e^{-\frac{L^2}{2}(\xi_i - \gamma_i)^2} + e^{-\frac{L^2}{2}(\xi_i + \gamma_i)^2} \right], \quad (2.12)$$

while (2.8) and (2.10) yield

$$\dot{A}(\xi_1, \xi_2, \xi_3, t_0) = 0. \quad (2.13)$$

An interesting generalization of the Cauchy data (2.9) and (2.10) might be taken to be

$$\Phi_0 \equiv e^{-\frac{(x^2+y^2+z^2)}{2L^2}} (\cos \gamma_1 x) (\cos \gamma_2 y) (\cos \gamma_3 z) (\cos \gamma t_0), \quad (2.14)$$

$$\Phi_1 \equiv \pm \gamma e^{-\frac{(x^2+y^2+z^2)}{2L^2}} (\cos \gamma_1 x) (\cos \gamma_2 y) (\cos \gamma_3 z) (\sin \gamma t_0), \quad (2.15)$$

since it reduces to (2.9) and (2.10) at  $t_0 = 0$ , which is indeed the value of initial time assumed in the Minkowski spacetime example considered in Ref. [8] (whereas in Kasner spacetime we take so far  $t_0 \neq 0$  to have enough equations to determine  $Z_1(\xi_1, \xi_2, \xi_3)$  and  $Z_2(\xi_1, \xi_2, \xi_3)$ ). Hereafter, to avoid cumbersome formulas, we keep choosing the Cauchy data (2.9) and (2.10). At this stage, Eqs. (2.7), (2.8), (2.12) and (2.13) lead to

$$Z_1(\xi_1, \xi_2, \xi_3) = \frac{\dot{Y}_{i\xi_1}}{(J_{i\xi_1} \dot{Y}_{i\xi_1} - Y_{i\xi_1} \dot{J}_{i\xi_1})} \Big|_{(t_0 \sqrt{(\xi_2)^2 + (\xi_3)^2}} A(\xi_1, \xi_2, \xi_3, t_0), \quad (2.16)$$

$$Z_2(\xi_1, \xi_2, \xi_3) = -\frac{\dot{J}_{i\xi_1}}{(J_{i\xi_1} \dot{Y}_{i\xi_1} - Y_{i\xi_1} \dot{J}_{i\xi_1})} \Big|_{(t_0 \sqrt{(\xi_2)^2 + (\xi_3)^2}} A(\xi_1, \xi_2, \xi_3, t_0), \quad (2.17)$$

where (2.12) should be used to express  $A(\xi_1, \xi_2, \xi_3, t_0)$ . The integrand of Eq. (1.4) is therefore expressed in factorized form through Bessel functions, decaying exponentials and oscillating functions, but the evaluation of the integral is hard, even in this simple case.

### 2.3. Representation of the solution through integration in the complex domain

Note now that the original hyperbolic equation (1.3) is a particular case of the general form [9]

$$L[u] \equiv \left[ \frac{\partial^2}{\partial t^2} - \left( \sum_{j,k=1}^n a^{jk} \frac{\partial^2}{\partial x^j \partial x^k} + b \frac{\partial}{\partial t} + \sum_{j=1}^n b^j \frac{\partial}{\partial x^j} + c \right) \right] u = 0. \quad (2.18)$$

In the general theory,  $a^{jk}$  is a symmetric tensor,  $b \frac{\partial}{\partial t} + \sum_{j=1}^n b^j \frac{\partial}{\partial x^j}$  is a  $C^\infty$  vector field and  $c$  is a  $C^\infty$  scalar field. In our case, we have  $n = 3$ ,  $x^1 = x$ ,  $x^2 = y$ ,  $x^3 = z$  and

$$a^{jk} = \text{diag}(t^{-2}, 1, 1), \quad b = \frac{1}{t}, \quad b^j = 0, \quad c = 0. \quad (2.19)$$

Thus, for all  $t \neq 0$  (as we said before, we avoid  $t = 0$ , which is a singularity of the Kasner coordinates), we can exploit the integral representation (see Appendix A) of the solution of hyperbolic equations with variable coefficients [9], while remarking that Eq. (1.3) is also of a type similar to other hyperbolic equations for which the mathematical literature (see Appendix A) has proved that the Cauchy problem is well posed [10,11]. On referring the reader to chapters 5 and 6 of Ref. [9] for the interesting details, we simply state here the main result when Eqs. (2.18) and (2.19) hold.

**Theorem 2.1** The solution of the scalar wave equation (1.3) with Cauchy data (2.9) and (2.10) at  $t = t_0 \neq 0$  admits the integral representation

$$u(t, x^1, x^2, x^3) = \lim_{\partial D \rightarrow T} \int_{\partial D} B[u(\tau, y^1, y^2, y^3), S(\tau, y^1, y^2, y^3; t, x^1, x^2, x^3)], \quad (2.20)$$

where  $S$  is a fundamental solution (see Appendix B) of the adjoint equation

$$M[S] = 0, \quad (2.21)$$

$M$  being the adjoint operator acting, in our case, as

$$M \equiv \frac{\partial^2}{\partial t^2} - \sum_{j,k=1}^3 a^{jk} \frac{\partial^2}{\partial x^j \partial x^k} + \frac{db}{dt} + b \frac{\partial}{\partial t}, \quad (2.22)$$

while the integrand  $B[u, S]$  is the differential 3-form

$$\begin{aligned} B[u, S] = & \left[ buS - \left( S \frac{\partial u}{\partial \tau} - u \frac{\partial S}{\partial \tau} \right) \right] dy^1 \wedge dy^2 \wedge dy^3 \\ & + \sum_{j=1}^3 (-1)^j \sum_{k=1}^3 a^{jk} \left( S \frac{\partial u}{\partial y^k} - u \frac{\partial S}{\partial y^k} \right) d\tau \wedge dy^1 \wedge \dots \widehat{dy^j} \wedge \dots dy^3. \end{aligned} \quad (2.23)$$

With this notation, the hat upon  $dy^j$  denotes omission of integration with respect to that particular variable, and  $D$  is the region of integration viewed as a cell in the complex domain, with boundary  $\partial D$ . Integration over  $\partial D$  should be therefore

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interpreted in the sense of the calculus of exterior differential forms. Our  $D$  is a manifold defined by the conditions

$$\text{Im}(\tau^2) + \sum_{k=1}^3 (y^k - x^k)^2 \leq \varepsilon^2, \quad (2.24)$$

$$\text{Re}(\tau - t) = \text{Im}(y^1) = \text{Im}(y^2) = \text{Im}(y^3) = 0, \quad (2.25)$$

which describe a sphere of radius  $\varepsilon$  in the complex domain, centered at the real point  $(t, x^1, x^2, x^3)$ . Moreover, the symbolic notation  $\partial D \rightarrow T$  indicates the process of describing the boundary  $\partial D$  down around the domain of dependence on the space where the initial data (2.9) and (2.10) are assigned (such a space is a two-dimensional plane when the Kasner exponents  $(1, 0, 0)$  are chosen, whereas, for more general exponents, it corresponds to a singular surface of infinite curvature). A quite complicated evaluation of residues is involved in Eq. (2.20), because the fundamental solution  $S$  of Eq. (2.21) is singular where the Hadamard-Ruse-Synge world function (see Appendix B) vanishes.

### 3. Bicharacteristics of the scalar wave equation in Kasner spacetime and the case $p_1 = p_2 = \frac{2}{3}, p_3 = -\frac{1}{3}$

Equation (1.3) is just a particular case of the following general form of scalar wave equation in Kasner spacetime:

$$P\phi = \left( -\frac{\partial^2}{\partial t^2} - \frac{1}{t} \frac{\partial}{\partial t} + \sum_{l=1}^3 t^{-2p_l} \frac{\partial^2}{\partial x^{l^2}} \right) \phi = 0. \quad (3.1)$$

In light of the technical results in Appendix A, it is rather important to study Eq. (3.1) with generic values of parameters, which is what we do now.

The leading symbol of the wave operator  $P$  is the contravariant form  $g^{\alpha\beta} = \text{diag}(-1, t^{-2p_1}, t^{-2p_2}, t^{-2p_3})$  of the metric, and hence the characteristic polynomial reads as

$$H(x, \xi) = g^{\alpha\beta}(x) \xi_\alpha \xi_\beta = -(\xi_0)^2 + \sum_{k=1}^3 t^{-2p_k} (\xi_k)^2. \quad (3.2)$$

An hypersurface  $\Sigma$  is called a *characteristic* of  $P$  if the restriction  $Pu|_\Sigma$  can be expressed by using only derivatives tangential to  $\Sigma$  of the restrictions  $u|_\Sigma$  and  $\text{grad}u|_\Sigma$ . This implies that the characteristics of wave equations on  $M$  are the null hypersurfaces of  $M$  [12]. Hence there is, at each point of the characteristic surface  $\Sigma$ , a unique null direction that is both normal and tangential to  $\Sigma$ . The curves on  $\Sigma$  that are tangential to this null direction field form a congruence on  $\Sigma$  and are called the *bicharacteristics*.

Suppose now that  $\Omega$  is a coordinate neighbourhood such that

$$\Omega \cap \Sigma = \{x : \chi(x) = 0\} \equiv \Sigma_\Omega, \quad (3.3)$$



where  $\chi \in C^\infty(\Omega)$  and  $\text{grad}\chi \neq 0$ . If  $\Sigma$  is a characteristic, one has [12,13]

$$\langle \text{grad}\chi, \text{grad}\chi \rangle = g^{\alpha\beta}(x) \frac{\partial\chi}{\partial x^\alpha} \frac{\partial\chi}{\partial x^\beta} = g^{-1}(\text{d}\chi, \text{d}\chi) = 0 \text{ when } \chi = 0. \quad (3.4)$$

By introducing  $\chi$  as a local coordinate, this implies, for some  $A \in C^\infty(\Omega)$ , the equation

$$g^{\alpha\beta}(x)(\partial_\alpha\chi)(\partial_\beta\chi) = A\chi. \quad (3.5)$$

The differential equations

$$\frac{d}{d\tau}x^\alpha = g^{\alpha\beta}(x)\partial_\beta\chi(x) \quad (3.6)$$

where  $\tau$  is a parameter, have a unique smooth solution for given initial values  $x^\alpha(0)$ . Since the gradient of  $\chi$  is orthogonal to a hypersurface  $\chi = \text{const}$ , the bicharacteristics of  $\Sigma$  are obtained by integrating Eq. (3.6) subject to the initial condition  $\chi(x(0)) = 0$ . If the map  $\tau \rightarrow x^\alpha(\tau)$  is an integral curve of Eq. (3.6), and

$$\pi_\alpha \equiv \partial_\alpha\chi(x(\tau)), \quad (3.7)$$

one finds

$$\frac{d}{d\tau}\pi_\alpha = (\partial_\alpha\partial_\beta\chi)\frac{d}{d\tau}x^\beta = (\partial_\alpha\partial_\beta\chi)g^{\beta\gamma}\partial_\gamma\chi = \frac{1}{2}\partial_\alpha\langle \text{grad}\chi, \text{grad}\chi \rangle - \frac{1}{2}\pi_\beta\pi_\gamma\partial_\alpha g^{\beta\gamma}. \quad (3.8)$$

On a bicharacteristic, one has  $\chi = 0 \implies \partial_\alpha(A\chi) = A\partial_\alpha\chi$ , and hence it follows from Eq. (3.5) that the equations

$$\frac{d}{d\tau}x^\alpha = g^{\alpha\beta}(x)\pi_\beta, \quad \frac{d}{d\tau}\pi_\alpha = -\frac{1}{2}\pi_\beta\pi_\gamma\partial_\alpha g^{\beta\gamma} + \frac{1}{2}A\pi_\alpha \quad (3.9)$$

hold on a bicharacteristic. One can introduce an invertible parameter transformation by defining a function  $\tau \rightarrow \psi(\tau)$  such that [12]

$$\frac{d\psi}{d\tau} = \exp\left(\frac{1}{2}\int_0^\tau A(x(\tau'))d\tau'\right), \quad \psi(\tau=0) = 0. \quad (3.10)$$

If one then defines

$$\xi_\alpha \equiv \frac{d\tau}{d\psi}\pi_\alpha = \frac{d\tau}{d\psi}\partial_\alpha\chi(x(\tau)), \quad (3.11)$$

the equations (3.9) are transformed into the equivalent Hamilton-like set

$$\frac{d}{d\psi}x^\alpha = g^{\alpha\beta}(x)\xi_\beta = \frac{1}{2} \frac{\partial H(x, \xi)}{\partial \xi_\alpha} \Big|_{H(x, \xi)|_{\Sigma_\Omega}}, \quad (3.12)$$

$$\frac{d}{d\psi}\xi_\alpha = -\frac{1}{2}\xi_\beta\xi_\gamma\partial_\alpha g^{\beta\gamma}(x) = -\frac{1}{2} \frac{\partial H(x, \xi)}{\partial x^\alpha} \Big|_{H(x, \xi)|_{\Sigma_\Omega}}, \quad (3.13)$$

where  $\frac{1}{2}H(x, \xi)$  plays the role of Hamiltonian function. Interestingly, Eqs. (3.12) and (3.13) are the equations of null geodesics, in canonical form. The null nature of such geodesics follows immediately from the definition (3.11) and Eq. (3.5), i.e.

$$H(x, \xi)|_{\Sigma_\Omega} = \left[ g^{\alpha\beta}(x) \xi_\alpha \xi_\beta \right]_{\Sigma_\Omega} = \left[ \left( \frac{d\tau}{d\psi} \right)^2 g^{\alpha\beta} (\partial_\alpha \chi) (\partial_\beta \chi) \right]_{\Sigma_\Omega} = \left[ \left( \frac{d\tau}{d\psi} \right)^2 A \chi \right]_{\Sigma_\Omega} = 0. \quad (3.14)$$

Thus, the bicharacteristics are null geodesics.

A characteristic contains a congruence of bicharacteristics, the latter being the null geodesics of spacetime  $(M, g)$ . The bicharacteristics of  $P$  can also be viewed as the projections on the manifold  $M$  of the curves in the cotangent bundle  $T^*M$ , called bicharacteristic strips, that are the integral curves of the Hamiltonian system of ordinary differential equations (3.12) and (3.13). The equation to which Eq. (3.5) reduces when  $\chi = 0$  (which occurs on characteristics and bicharacteristics) is the well known *eikonal equation* (cf. Ref. [14]). If the parameter  $\psi$  is set equal to  $2\zeta$ , there is complete formal analogy between Eqs. (3.12), (3.13) and a set of Hamilton equations. As far as wave propagation is concerned, the *wave fronts* of our wave equation (3.1) are characteristic surfaces (satisfying therefore Eqs. (3.3) and (3.5)) and propagate along bicharacteristics [9,15]. With our Kasner metric and  $\psi = 2\zeta$ , the Eqs. (3.12) and (3.13) for bicharacteristics, bearing in mind that  $x^0 = t$  in  $c = 1$  units, take the form (there is no summation over  $i$  in Eq. (3.16) below)

$$\frac{dt}{d\zeta} = -2\xi_0(\zeta), \quad (3.15)$$

$$\frac{dx^i}{d\zeta} = 2\xi_i(t(\zeta))^{-2p_i}, \quad (3.16)$$

$$\frac{d\xi_0}{d\zeta} = 2 \sum_{k=1}^3 p_k (\xi_k)^2 (t(\zeta))^{-2p_k-1}, \quad (3.17)$$

$$\frac{d\xi_i}{d\zeta} = 0. \quad (3.18)$$

Equation (3.18) is solved by

$$\xi_i = \tilde{\xi}_i \text{ constant } \forall i = 1, 2, 3. \quad (3.19)$$

Further differentiation with respect to  $\zeta$  of Eq. (3.15) leads therefore, by virtue of (3.17), to the equivalent decoupled system given by

$$\frac{d^2 t}{d\zeta^2} = -4 \sum_{k=1}^3 p_k (\tilde{\xi}_k)^2 (t(\zeta))^{-2p_k-1} \quad (3.20)$$

together with Eqs. (3.16) and (3.17). Upon defining  $A_k \equiv -4p_k(\tilde{\xi}_k)^2$  we find, more explicitly, the following nonlinear equation for  $t(\zeta)$ :

$$\frac{d^2 t}{d\zeta^2} = \frac{A_1}{t^{2p_1+1}} + \frac{A_2}{t^{2p_2+1}} + \frac{A_3}{t^{2p_3+1}} = f(t), \quad (3.21)$$

supplemented by the initial conditions

$$t'(\zeta = 0) = -2\xi_0(0), \quad t(\zeta = 0) = t_0. \quad (3.22)$$

Since the right-hand side of Eq. (3.21) is independent of  $\zeta$  we are dealing with an autonomous differential equation, which can be solved by separation of variables. For this purpose, setting  $t' = \frac{dt}{d\zeta} = Q(t)$  one finds

$$t'' = \frac{dQ}{dt} \frac{dt}{d\zeta} = Q \frac{dQ}{dt} = f(t) \implies Q dQ = f(t) dt, \quad (3.23)$$

which implies, denoting by  $\gamma$  an integration constant,

$$Q^2 = \gamma + 2 \int f(t) dt, \quad (3.24)$$

and hence

$$Q = \pm \sqrt{\gamma + 2 \int f(t) dt} = \frac{dt}{d\zeta}, \quad (3.25)$$

where a further integration yields

$$\int \frac{dt}{\sqrt{\gamma + 2 \int f(t) dt}} = \pm \zeta + \kappa, \quad (3.26)$$

for some constant  $\kappa$ . In our case, the integral of  $f$  is such that

$$2 \int f(t) dt = 2 \sum_{k=1}^3 -\frac{A_k}{2p_k} t^{-2p_k} = \sum_{k=1}^3 4(\tilde{\xi}_k)^2 t^{-2p_k}, \quad (3.27)$$

so that Eq. (3.26) for the geodesic parameter  $\zeta$  in terms of the time variable  $t$  reads as

$$\pm \zeta(t) = \int \frac{dt}{\sqrt{\gamma + \sum_{k=1}^3 4(\tilde{\xi}_k)^2 t^{-2p_k}}} - \kappa. \quad (3.28)$$

As far as we can see, such an integral cannot be evaluated explicitly for generic values of Kasner parameters, but in the particular case

$$p_1 = p_2 = \frac{2}{3}, \quad p_3 = -\frac{1}{3}, \quad (3.29)$$

which is technically harder than the  $(1, 0, 0)$  choice studied in Ref. [16], we have found an explicit formula in terms of elliptic integrals. To be self-contained, we recall some basic definitions, as follows.

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(i) If  $\Omega \in ]-\frac{\pi}{2}, \frac{\pi}{2}[$ , the elliptic integral of the first kind, here denoted by  $E_I[\Omega, m]$ , is defined by

$$E_I[\Omega, m] \equiv \int_0^\Omega (1 - m \sin^2 \theta)^{-\frac{1}{2}} d\theta. \quad (3.30)$$

(ii) If  $\Omega \in ]-\frac{\pi}{2}, \frac{\pi}{2}[$ , the elliptic integral of the second kind,  $E_{II}[\Omega, m]$ , reads as

$$E_{II}[\Omega, m] \equiv \int_0^\Omega (1 - m \sin^2 \theta)^{\frac{1}{2}} d\theta. \quad (3.31)$$

(iii) If  $\Omega$  lies the same open interval as in (i) and (ii), the incomplete elliptic integral of the third kind is given by

$$E_{III}[n, \Omega, m] \equiv \int_0^\Omega (1 - n \sin^2 \theta)^{-1} (1 - m \sin^2 \theta)^{-\frac{1}{2}} d\theta. \quad (3.32)$$

Moreover, in order to express our results, we have to consider the three roots  $r_1, r_2, r_3$  of the algebraic equation

$$x^3 + \frac{\gamma}{4(\tilde{\xi}_3)^2} x^2 + \frac{[(\tilde{\xi}_1)^2 + (\tilde{\xi}_2)^2]}{(\tilde{\xi}_3)^2} = 0. \quad (3.33)$$

If the three roots are all real, we follow the convention according to which  $r_1 < r_2 < r_3$ . If  $r_1$  is the real root while the remaining two roots are complex conjugate, we agree that  $r_2$  and  $r_3$  are such that  $\text{Im}r_2 < \text{Im}r_3$ . With this understanding, and defining

$$f_{k,1}(r) \equiv \frac{r_k}{(r_k - r_1)}, k = 2, 3, \quad (3.34)$$

$$f(r, t) \equiv \sqrt{\frac{t^{2/3}}{f_{3,1}(r)(t^{2/3} - r_1)}}, \quad (3.35)$$

$$\begin{aligned} H(\xi, r, t) &\equiv 4((\tilde{\xi}_1)^2 + (\tilde{\xi}_2)^2) \sqrt{\frac{r_1(t^{2/3} - r_2)}{r_2(t^{2/3} - r_1)}} \\ &\times \left\{ 4(\tilde{\xi}_3)^2 \left( \frac{r_3}{r_1} - 1 \right) E_{II}[\arcsin f(r, t), f_{3,1}(r)/f_{2,1}(r)] \right. \\ &+ \frac{1}{r_2} \left[ (\gamma + 4(\tilde{\xi}_3)^2 r_2) E_I[\arcsin f(r, t), f_{3,1}(r)/f_{2,1}(r)] \right. \\ &\left. \left. - \gamma E_{III}[f_{3,1}(r), \arcsin f(r, t), f_{3,1}(r)/f_{2,1}(r)] \right] \right\}, \end{aligned} \quad (3.36)$$

the integral (3.28) reads as

$$\begin{aligned} \pm \zeta(t) &= \frac{1}{2} \left( \frac{4((\tilde{\xi}_1)^2 + (\tilde{\xi}_2)^2) + \gamma t^{\frac{4}{3}} + 4(\tilde{\xi}_3)^2 t^2}{t^{2/3}} \right)^{-1/2} \left[ 3(t^{2/3} - r_3) \right. \\ &\times \left. \left( t^{2/3} - r_2 - \frac{H(\xi, r, t)(t^{2/3} - r_1)}{16(\tilde{\xi}_3)^4 (r_3)^2 \sqrt{\frac{r_1(t^{2/3} - r_3)}{r_3 f_{3,1}(r)}}} \right) \right]. \end{aligned} \quad (3.37)$$

Furthermore, since Eq. (3.37) cannot be explicitly inverted to express  $t = t(\zeta)$ , it is more convenient to use the identities

$$\frac{dx^i}{dt} = \frac{dx^i}{d\zeta} \frac{d\zeta}{dt}, \quad \frac{d\xi_0}{dt} = \frac{d\xi_0}{d\zeta} \frac{d\zeta}{dt}, \quad (3.38)$$

so that the remaining components of null geodesic equations are expressed, from (3.26) and (3.38), in the form

$$\int dx^i = \pm 2\tilde{\xi}_i \int \frac{t^{-2p_i}}{\sqrt{\gamma + \sum_{k=1}^3 4(\tilde{\xi}_k)^2 t^{-2p_k}}} dt, \quad (3.39)$$

$$\int d\xi_0 = \pm 2 \int \frac{\sum_{k=1}^3 p_k (\tilde{\xi}_k)^2 t^{-2p_k-1}}{\sqrt{\gamma + \sum_{k=1}^3 4(\tilde{\xi}_k)^2 t^{-2p_k}}} dt. \quad (3.40)$$

With the choice (3.29) of Kasner parameters, also (3.39) can be expressed in terms of elliptic integrals, while (3.40) takes the remarkably simple form

$$\xi_0(t) = \mp \frac{1}{2} \sqrt{\frac{4[(\tilde{\xi}_1)^2 + (\tilde{\xi}_2)^2] + \gamma t^{\frac{4}{3}} + 4(\tilde{\xi}_3)^2 t^2}{t^{\frac{4}{3}}}}. \quad (3.41)$$

In the formula (3.39), when  $i = 1, 2$ , we find

$$\frac{\int dx^1}{2\tilde{\xi}_1} = \frac{\int dx^2}{2\tilde{\xi}_2} = \pm I(t), \quad (3.42)$$

where

$$I(t) = \mp \frac{3}{2} \frac{t^{-2/3}}{\xi_0} E_I[\arcsin f(r, t), f_{3,1}(r)/f_{2,1}(r)] \sqrt{(t^{2/3} - r_1)(t^{2/3} - r_2)(t^{2/3} - r_3) \frac{f_{3,1}(r)}{r_2 r_3}}, \quad (3.43)$$

while, if  $i = 3$ , we find

$$\frac{\int dx^3}{2\tilde{\xi}_3} = \pm I_3(t), \quad (3.44)$$

where, upon defining

$$A^2 \equiv 4(\tilde{\xi}_1)^2, \quad B^2 \equiv 4(\tilde{\xi}_2)^2, \quad C^2 \equiv 4(\tilde{\xi}_3)^2, \quad \rho(t) \equiv \sqrt{A^2 + B^2 + \gamma t^{4/3} + C^2 t^2}, \quad (3.45)$$

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we write

$$\begin{aligned}
I_3(t) \equiv & \frac{3}{4C^2} \sqrt{\rho(t)} t^{1/3} - \frac{3}{8C^2} \left\{ - \left[ \left( 2(A^2 + B^2)r_3(t^{2/3} - r_1)^2 \right. \right. \right. \\
& \times \frac{\sqrt{(t^{2/3} - r_2)(t^{2/3} - r_3)}}{(t^{2/3} - r_1)^{3/2}(r_3 - r_1)\sqrt{\rho(t)}r_2r_3} E_I[\arcsinf(r, t), f_{3,1}(r)/f_{2,1}(r)] \\
& + 3\gamma \left( (t^{2/3} - r_2)(t^{2/3} - r_3)t^{2/3} \right. \\
& + r_3(t^{2/3} - r_1)^2 \frac{\sqrt{(t^{2/3} - r_2)(t^{2/3} - r_3)}r_1}{(t^{2/3} - r_1)^{3/2}\sqrt{\rho(t)}r_2r_3} \\
& \times \left( \frac{r_1 + r_3}{r_1 - r_3} E_I[\arcsinf(r, t), f_{3,1}(r)/f_{2,1}(r)] + \frac{r_2}{r_1} E_{II}[\arcsinf(r, t), f_{3,1}(r)/f_{2,1}(r)] \right. \\
& \left. \left. \left. + \frac{(r_1 + r_2 + r_3)}{(r_3 - r_1)} E_{III}[f_{3,1}(r), \arcsinf(r, t), f_{3,1}(r)/f_{2,1}(r)] \right) \right] \right\}. \quad (3.46)
\end{aligned}$$

Our original calculation of bicharacteristics in Kasner spacetime is therefore completed.

#### 4. Parametrix for the Cauchy problem through Fourier-Maslov integral operators: the general case

Since we are studying a wave equation, we may expect that the solution formula involves amplitude and phase functions, as well as the Cauchy data (here, unlike Sec. II, we exploit techniques that do not need to avoid  $t = 0$ )

$$\phi(t = 0, x) \equiv u_0(x), \quad (4.1)$$

$$\frac{\partial \phi}{\partial t}(t = 0, x) \equiv u_1(x), \quad (4.2)$$

which are again assumed to be Fourier transformable. However, the variable nature of the coefficients demands for a nontrivial generalization of the integral representation (1.4). This is indeed available, since a theorem guarantees that the solution of the Cauchy problem (3.1), (4.1) and (4.2) can be expressed in the form [3]

$$\phi(x, t) = \sum_{j=0}^1 E_j(t) u_j(x), \quad (4.3)$$

where, on denoting by  $\hat{u}_j$  the Fourier transform of the Cauchy data, the operators  $E_j(t)$  act according to (hereafter,  $(x) \equiv (x^1, x^2, x^3)$ , with covariable  $(\xi) \equiv (\xi_1, \xi_2, \xi_3)$ )

$$E_j(t) u_j(x) = \sum_{k=1}^2 (2\pi)^{-3} \int e^{i\varphi_k(x, t, \xi)} \alpha_{jk}(x, t, \xi) \hat{u}_j(\xi) d^3 \xi + R_j(t) u_j(x), \quad (4.4)$$

where the  $\varphi_k$  are real-valued phase functions which satisfy the initial condition

$$\varphi_k(t=0, x, \xi) = x \cdot \xi = \sum_{s=1}^3 x^s \xi_s, \quad (4.5)$$

and  $R_j(t)$  is a regularizing operator which smoothes out the singularities acted upon by it [3]. In other words, the Cauchy problem is here solved by a pair of Fourier-Maslov integral operators of the form (4.4), and such a construction (leaving aside, for the moment, its global version, which can be built as shown in Chapter VII of Ref. [3]) generalizes the monochromatic plane waves for the d'Alembert operator from Minkowski spacetime to Kasner spacetime. Strictly, we are dealing with the approximate Green function for the wave equation, called the parametrix. In our case, since we know a priori that (4.3) and (4.4) yield an exact solution of the Cauchy problem, we can insert them into Eq. (3.1), finding that, for all  $j = 0, 1$ ,

$$P[E_j(t)u_j(x)] \sim \sum_{k=1}^2 (2\pi)^{-3} \int P[e^{i\varphi_k} \alpha_{jk}] \hat{u}_j(\xi) d^3\xi, \quad (4.6)$$

where  $PR_j(t)u_j(x)$  can be neglected with respect to the integral on the right-hand side of Eq. (4.4), because  $R_j(t)$  is a regularizing operator. Next, we find from Eq. (3.1) that

$$P[e^{i\varphi_k} \alpha_{jk}] = e^{i\varphi_k} (iA_{jk} + B_{jk}), \quad (4.7)$$

where

$$A_{jk} \equiv \frac{\partial^2 \varphi_k}{\partial t^2} \alpha_{jk} + 2 \frac{\partial \varphi_k}{\partial t} \frac{\partial \alpha_{jk}}{\partial t} + \frac{1}{t} \frac{\partial \varphi_k}{\partial t} \alpha_{jk} - \sum_{l=1}^3 t^{-2p_l} \left( \frac{\partial^2 \varphi_k}{\partial x_l^2} \alpha_{jk} + 2 \frac{\partial \varphi_k}{\partial x_l} \frac{\partial \alpha_{jk}}{\partial x_l} \right), \quad (4.8)$$

$$B_{jk} \equiv \frac{\partial^2 \alpha_{jk}}{\partial t^2} - \left( \frac{\partial \varphi_k}{\partial t} \right)^2 \alpha_{jk} + \frac{1}{t} \frac{\partial \alpha_{jk}}{\partial t} - \sum_{l=1}^3 t^{-2p_l} \left( \frac{\partial^2 \alpha_{jk}}{\partial x_l^2} - \left( \frac{\partial \varphi_k}{\partial x_l} \right)^2 \alpha_{jk} \right). \quad (4.9)$$

If the phase functions  $\varphi_k$  are real-valued, since the exponentials  $e^{i\varphi_k}$  can be taken to be linearly independent, we can fulfill Eq. (3.1), up to the negligible contributions resulting from  $PR_j(t)u_j(x)$ , by setting to zero in the integrand (4.6) both  $A_{jk}$  and  $B_{jk}$ . This leads to a coupled system of partial differential equations. Our Cauchy problem (3.1), (4.1) and (4.2) is therefore equivalent to solving the equations

$$A_{jk} = 0, \quad B_{jk} = 0. \quad (4.10)$$

Equation (4.10) is the *dispersion relation* for the scalar wave equation in Kasner spacetime. Such a dispersion relation takes a neater geometric form upon bearing in mind the form (3.1) of the wave (or d'Alembert) operator  $P = \square$  in Kasner coordinates, i.e.

$$A_{jk} = 0 \implies \left[ -\alpha_{jk}(\square \varphi_k) - 2g^{\beta\gamma}(\varphi_k)_{,\beta}(\alpha_{jk})_{,\gamma} \right] = 0, \quad (4.11)$$

$$B_{jk} = 0 \implies \left[ -\square + g^{\beta\gamma}(\varphi_k)_{,\beta}(\varphi_k)_{,\gamma} \right] \alpha_{jk} = 0. \quad (4.12)$$

Let us bear in mind that the indices  $j$  and  $k$  are not tensorial, but they merely count the number of functions contributing to the Fourier-Maslov integral operator (4.4). We can therefore exploit the four-dimensional concept of gradient of a function [14] as the four-dimensional covariant vector defined by the differential of the function, i.e.

$$df = \frac{\partial f}{\partial x^\alpha} dx^\alpha = f_{,\alpha} dx^\alpha = (\nabla_\alpha f) dx^\alpha = (\text{grad} f)_\alpha dx^\alpha, \quad (4.13)$$

where  $\nabla$  is the Levi-Civita connection on four-dimensional spacetime, and we exploit the identity  $f_{,\alpha} = \nabla_\alpha f$ ,  $\forall f \in C^\infty(M)$ . The consideration of  $\nabla_\alpha f$  is not mandatory at this stage, but it will be helpful in a moment, when we write (see below) in tensor language the equations expressing the dispersion relation.

We arrive therefore, upon multiplying Eq. (4.11) by  $\alpha_{jk}$ , while dividing Eq. (4.12) by  $\alpha_{jk}$ , at the following geometric form of dispersion relation in Kasner spacetime (with our notation we actually write it in the same way in any Lorentzian spacetime):

$$g^{\beta\gamma} \nabla_\beta \left[ (\alpha_{jk})^2 \nabla_\gamma \varphi_k \right] = \text{div} \left[ (\alpha_{jk})^2 \text{grad} \varphi_k \right] = 0, \quad (4.14)$$

$$g^{\beta\gamma} (\nabla_\beta \varphi_k) (\nabla_\gamma \varphi_k) = \langle \text{grad} \varphi_k, \text{grad} \varphi_k \rangle = \frac{(\square \alpha_{jk})}{\alpha_{jk}}, \quad (4.15)$$

where the four-dimensional divergence operator acts according to<sup>d</sup>

$$\text{div} F = \nabla^\beta F_\beta = g^{\alpha\beta} \nabla_\alpha F_\beta. \quad (4.16)$$

Note that, if the ratio  $\frac{(\square \alpha_{jk})}{\alpha_{jk}}$  is much smaller than a suitable parameter having dimension  $\text{length}^{-2}$ , Eq. (4.15) reduces to the eikonal equation and hence the phase function reduces to the Hadamard-Ruse-Synge world function of Appendices B and C. This property makes contact with the asymptotic expansion presented in Appendix B. However, it is possible to devise a strategy to solve exactly Eqs. (4.14) and (4.15). For this purpose we remark that, upon defining the covariant vector

$$\psi_\gamma \equiv (\alpha_{jk})^2 \nabla_\gamma \varphi_k, \quad (4.17)$$

Eq. (4.14) is equivalent to solving the first-order partial differential equation expressing the vanishing divergence condition for  $\psi_\gamma$ , i.e.

$$\nabla^\gamma \psi_\gamma = \text{div} \psi = 0. \quad (4.18)$$

<sup>d</sup>In particular, when  $F$  is a gradient, one gets therefore the wave operator on scalars, i.e.

$$\square \equiv \text{div grad} = g^{\alpha\beta} \nabla_\alpha \nabla_\beta.$$



Of course, this equation is not enough to determine the four components of  $\psi_\gamma$ , but there are cases where further progress can be made (see below). After doing that, we can express the (covariant) derivative of the phase function from the definition (4.17), i.e.

$$\nabla_\gamma \varphi_k = \partial_\gamma \varphi_k = (\alpha_{jk})^{-2} \psi_\gamma, \quad (4.19)$$

and the insertion of Eq. (4.19) into Eq. (4.15) yields

$$(\alpha_{jk})^3 \square \alpha_{jk} = g(\psi, \psi) = g^{\beta\gamma} \psi_\beta \psi_\gamma = \psi_\gamma \psi^\gamma. \quad (4.20)$$

Interestingly, this is a tensorial generalization of a famous nonlinear ordinary differential equation, i.e. the Ermakov-Pinney equation [17,18,19]

$$y'' + py = qy^{-3}. \quad (4.21)$$

If  $y''$  is replaced by  $\square y$ ,  $p$  is set to zero and  $q$  is promoted to a function of spacetime location, Eq. (4.21) is mapped into Eq. (4.20). After solving this nonlinear equation for  $\alpha_{jk} = \alpha_{jk}[g(\psi, \psi)]$ , the task remains of finding the phase function  $\varphi_k$  by writing and solving the four components of Eq. (4.19). To sum up, we have proved the following original result.

**Theorem 4.1** For any Lorentzian spacetime manifold  $(M, g)$ , the amplitude functions  $\alpha_{jk} \in C^2(T^*M)$  and phase functions  $\varphi_k \in C^1(T^*M)$  in the parametrix (4.4) for the scalar wave equation can be obtained by solving, first, the linear condition (4.18) of vanishing divergence for a covariant vector  $\psi_\gamma$ . All nonlinearities of the coupled system are then mapped into solving the nonlinear equation (4.20) for the amplitude function  $\alpha_{jk}$ . Eventually, the phase function  $\varphi_k$  is found by solving the first-order linear equation (4.19).

In Kasner spacetime, Eq. (4.18) takes indeed the form

$$\left( \frac{\partial}{\partial t} + \frac{1}{t} \right) \psi_0 = \sum_{l=1}^3 t^{-2p_l} \frac{\partial \psi_l}{\partial x^l}. \quad (4.22)$$

This suggests considering  $\tilde{\psi}_0$  and  $\tilde{\psi}_l$  such that

$$\psi_0 = \frac{1}{t} \tilde{\psi}_0, \quad \psi_l = t^{2p_l-1} \tilde{\psi}_l \quad \forall l = 1, 2, 3, \quad (4.23)$$

so that Eq. (4.22) leads to the equation

$$\frac{\partial \tilde{\psi}_0}{\partial t} = \sum_{l=1}^3 \frac{\partial \tilde{\psi}_l}{\partial x^l}. \quad (4.24)$$

This is precisely the vanishing divergence condition satisfied by retarded potentials in Minkowski spacetime in the coordinates  $(t, x, y, z)$ . Their integral representation is well known to be of the form (recall that we work in  $c = 1$  units)

$$\tilde{\psi}_0 = \int \int \int \frac{\rho(t - r(x, y, z; x', y', z'))}{r(x, y, z; x', y', z')} dx' dy' dz', \quad (4.25)$$

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$$\tilde{\psi}_l = \int \int \int \frac{s_l(t - r(x, y, z; x', y', z'))}{r(x, y, z; x', y', z')} dx' dy' dz', \quad (4.26)$$

where

$$r \equiv \sqrt{(x - x')^2 + (y - y')^2 + (z - z')^2}, \quad (4.27)$$

and hence Eqs. (4.23), (4.25) and (4.26) solve completely the problem of finding the auxiliary covariant vector  $\psi_\gamma$  in Kasner spacetime.

We should now solve Eq. (4.20) for  $\alpha_{jk}$ . The reader might wonder what has been gained by turning the task of solving the scalar wave equation into the task of solving Eq. (4.20). In this equation, we can first get rid of the part linear in  $\frac{\partial}{\partial t}$  in the  $\square$  operator by setting

$$\alpha_{jk} = \frac{1}{\sqrt{t}} \tilde{\alpha}_{jk}, \quad (4.28)$$

which leads to

$$(\tilde{\alpha}_{jk})^3 \left[ -\frac{\partial^2}{\partial t^2} - \frac{1}{4t^2} + \sum_{l=1}^3 t^{-2p_l} \frac{\partial^2}{\partial x^{l2}} \right] \tilde{\alpha}_{jk} = t^2 \psi_\gamma \psi^\gamma. \quad (4.29)$$

Next, we can get rid of powers of  $\tilde{\alpha}_{jk}$  by setting  $\tilde{\alpha}_{jk} \equiv (f_{jk})^\beta$ , which yields

$$(\tilde{\alpha}_{jk})^3 \frac{\partial^2 \tilde{\alpha}_{jk}}{\partial t^2} = \beta (f_{jk})^{4\beta-1} \frac{\partial^2 f_{jk}}{\partial t^2} + \beta(\beta-1) (f_{jk})^{4\beta-2} \left( \frac{\partial f_{jk}}{\partial t} \right)^2, \quad (4.30)$$

and the same formula holds with  $t$  replaced by  $x^l$ . Thus, upon choosing  $\beta = \frac{1}{4}$ , we obtain eventually the amplitude functions by solving the following nonlinear equation for  $f_{jk}$ :

$$\begin{aligned} Lf_{jk} &\equiv \left( \frac{\partial^2}{\partial t^2} + \frac{1}{t^2} - \sum_{l=1}^3 t^{-2p_l} \frac{\partial^2}{\partial x^{l2}} \right) f_{jk} \\ &= \frac{3}{4} (f_{jk})^{-1} \left[ \left( \frac{\partial f_{jk}}{\partial t} \right)^2 - \sum_{l=1}^3 t^{-2p_l} \left( \frac{\partial f_{jk}}{\partial x^l} \right)^2 \right] + 4t^2 \left[ (\psi_0)^2 - \sum_{l=1}^3 t^{-2p_l} (\psi_l)^2 \right] \end{aligned} \quad (4.31)$$

The form (4.31) of the equation for  $f_{jk}$  makes it possible to apply the powerful Adomian method [20] for the solution of nonlinear partial differential equations. For this purpose, inspired by Ref. [20], we define the four linear operators occurring in  $L$ , i.e.

$$L_t \equiv \frac{\partial^2}{\partial t^2}, \quad L_x \equiv t^{-2p_1} \frac{\partial^2}{\partial x^2}, \quad L_y \equiv t^{-2p_2} \frac{\partial^2}{\partial y^2}, \quad L_z \equiv t^{-2p_3} \frac{\partial^2}{\partial z^2}, \quad (4.32)$$

the remainder (i.e., lower order part) of the linear operator  $L$ , i.e.

$$R \equiv \frac{1}{t^2}, \quad (4.33)$$

the nonlinear term (hereafter we omit the subscripts  $j, k$  for simplicity of notation)

$$Nf \equiv \frac{3}{4}f \left[ \sum_{l=1}^3 t^{-2p_l} \left( \frac{1}{f} \frac{\partial f}{\partial x^l} \right)^2 - \left( \frac{1}{f} \frac{\partial f}{\partial t} \right)^2 \right], \quad (4.34)$$

while the part of the right-hand side which is independent of  $f$  is here denoted by  $\eta$ , i.e.

$$\eta \equiv 4t^2 \left[ (\psi_0)^2 - \sum_{l=1}^3 t^{-2p_l} (\psi_l)^2 \right]. \quad (4.35)$$

Hence the nonlinear equation (4.31) can be re-expressed in the form

$$(L_t - L_x - L_y - L_z)f + (R + N)f = \eta. \quad (4.36)$$

The idea is now to apply the inverse of  $L_t$ , or  $L_x$ , or  $L_y$ , or  $L_z$  to this equation, which, upon bearing in mind the identities [20]

$$L_t^{-1} L_t f = f - \alpha_1 - \alpha_2 t, \quad (4.37)$$

$$L_x^{-1} L_x f = f - \alpha_3 - \alpha_4 x, \quad (4.38)$$

$$L_y^{-1} L_y f = f - \alpha_5 - \alpha_6 y, \quad (4.39)$$

$$L_z^{-1} L_z f = f - \alpha_7 - \alpha_8 z, \quad (4.40)$$

the  $\alpha$ 's being constants fixed by the initial and boundary conditions, leads to the following four equations:

$$f = \alpha_1 + \alpha_2 t + L_t^{-1}(L_x + L_y + L_z)f - L_t^{-1}Rf - L_t^{-1}Nf + L_t^{-1}\eta, \quad (4.41)$$

$$f = \alpha_3 + \alpha_4 x + L_x^{-1}(L_t - L_y - L_z)f + L_x^{-1}Rf + L_x^{-1}Nf - L_x^{-1}\eta, \quad (4.42)$$

$$f = \alpha_5 + \alpha_6 y + L_y^{-1}(L_t - L_z - L_x)f + L_y^{-1}Rf + L_y^{-1}Nf - L_y^{-1}\eta, \quad (4.43)$$

$$f = \alpha_7 + \alpha_8 z + L_z^{-1}(L_t - L_x - L_y)f + L_z^{-1}Rf + L_z^{-1}Nf - L_z^{-1}\eta. \quad (4.44)$$

Now we add these four equations, and upon defining

$$\begin{aligned} K \equiv & \frac{1}{4} \left[ L_t^{-1}(L_x + L_y + L_z - R) + L_x^{-1}(L_t - L_y - L_z + R) \right. \\ & \left. + L_y^{-1}(L_t - L_z - L_x + R) + L_z^{-1}(L_t - L_x - L_y + R) \right], \end{aligned} \quad (4.45)$$

$$G \equiv \frac{1}{4}(L_t^{-1} - L_x^{-1} - L_y^{-1} - L_z^{-1}), \quad (4.46)$$

$$f_0 \equiv \frac{1}{4}[\alpha_1 + \alpha_2 t + \alpha_3 + \alpha_4 x + \alpha_5 + \alpha_6 y + \alpha_7 + \alpha_8 z] + G\eta, \quad (4.47)$$

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we arrive at the fundamental formula

$$f = f_0 + Kf - GNf. \quad (4.48)$$

At this stage, if the function  $f$  has a Poincaré asymptotic expansion [21], which can be convergent or divergent and is written in the form

$$f \sim \sum_{n=0}^{\infty} f_n, \quad \frac{f_p}{f_0} < 1 \quad \forall p = 1, 2, \dots, \infty, \quad (4.49)$$

we point out that (4.49) leads in turn to a Poincaré asymptotic expansion of the nonlinear term  $Nf$  defined in Eq. (4.34) in the form

$$Nf \sim \sum_{n=0}^{\infty} A_n = A_0[f_0] + A_1[f_0, f_1] + \dots + A_k[f_0, \dots, f_k] + \dots, \quad (4.50)$$

where, by virtue of the formula

$$\log(1 + \omega) \sim \omega - \frac{\omega^2}{2} + \frac{\omega^3}{3} + \dots = \lim_{N \rightarrow +\infty} \sum_{k=0}^N (-1)^k \frac{\omega^{k+1}}{k} \quad \text{as } \omega \rightarrow 0, \quad (4.51)$$

we can evaluate the Poincaré asymptotic expansion of squared logarithmic derivatives according to

$$\begin{aligned} \left( \frac{1}{f} \frac{\partial f}{\partial x^l} \right)^2 &= \left( \frac{f_{,l}}{f} \right)^2 = \left\{ \left[ \log f_0 + \log \left( 1 + \frac{f_1}{f_0} + \frac{f_2}{f_0} + \dots \right) \right]_{,l} \right\}^2 \\ &\sim \left[ \frac{(f_{0,l} + f_{1,l})}{f_0} - \frac{f_1 f_{0,l}}{(f_0)^2} - \frac{f_1 f_{1,l}}{(f_0)^2} + \frac{(f_1)^2 f_{0,l}}{(f_0)^3} + \dots \right]^2. \end{aligned} \quad (4.52)$$

In light of (4.34) and (4.52) we find

$$A_0[f_0] = \frac{3}{4f_0} \left[ \sum_{l=1}^3 t^{-2p_l} (f_{0,l})^2 - (f_{0,t})^2 \right], \quad (4.53)$$

$$A_1[f_0, f_1] = \frac{3}{4f_0} \left( f_1 \left( 1 - \frac{f_1}{f_0} \right) \right)^2 \left\{ \sum_{l=1}^3 t^{-2p_l} \left[ \frac{f_{1,l}}{f_1} - \frac{f_{0,l}}{f_0} \right]^2 - \left[ \frac{f_{1,t}}{f_1} - \frac{f_{0,t}}{f_0} \right]^2 \right\}, \quad (4.54)$$

plus a countable infinity of other formulas for  $A_2[f_0, f_1, f_2], \dots, A_k[f_0, f_1, \dots, f_k] \dots$ . Note that, unlike the case of simpler nonlinearities [20], the functionals  $A_n$  involve division by  $f_0, f_1, \dots$ . The solution algorithm is now completely specified, because Eq. (4.48) yields the recursive formulas [20]

$$f_{n+1} = Kf_n - GA_n[f_0, \dots, f_n] \quad \forall n = 0, 1, \dots, \infty, \quad (4.55)$$

and hence

$$\sum_{p=0}^n f_p = (I + K + \dots + K^n) f_0 - (I + K + \dots + K^{n-1}) G A_0 - \dots - (I + \dots + K^{n-2}) G A_{n-2} - G A_{n-1}, \quad (4.56)$$

where, by exploiting the partial sum of the geometric series, we find

$$f \sim \lim_{n \rightarrow \infty} \sum_{p=0}^n f_p = \lim_{n \rightarrow \infty} \left[ \frac{(I - K^{n+1})}{(I - K)} f_0 - \frac{(I - K^n)}{(I - K)} GA_0 - \dots - GA_{n-1} \right]. \quad (4.57)$$

Since the operator  $K$  is built from the inverses of differential operators, it is a pseudo-differential operator, and it remains to be seen whether, for sufficiently large values of  $n$ , it only contributes to the terms  $R_j(t)$  in the parametrix (4.4), so that we only need the limit

$$\lim_{n \rightarrow \infty} \left[ (I - K)^{-1} (f_0 - GA_0) - \dots - GA_{n-1} \right]. \quad (4.58)$$

The Adomian method we have used is well suited to go beyond weak nonlinearity and small perturbations, but of course the nontrivial technical problem is whether the series for the unknown function  $f$  is convergent, and also how fast. If it were necessary to consider hundreds of terms, the algorithm would be of little practical utility.

An interesting alternative, which cannot be ruled out at present, is instead the existence of an asymptotic expansion of  $f$  involving only finitely many terms, whose rigorous theory is described in a monograph by Dieudonné [22]. In such a case we might write

$$f \sim f_0 + f_1 + f_2 = (I + K + K^2)f_0 - (I + K)GA_0[f_0] - GA_1[f_0, f_1 = Kf_0 - GA_0[f_0]], \quad (4.59)$$

which is fully computable by virtue of Eqs. (4.45)-(4.47) and (4.53)-(4.55). We find it therefore encouraging that an exact solution algorithm has been obtained for the scalar parametrix in Kasner spacetime.

Last, but not least, Eqs. (4.19) for the gradient of phase functions can be integrated to find

$$\begin{aligned} \varphi_k &= \int (\alpha_{jk})^{-2} \psi_0 dt + \Phi_{0,k}(x, y, z, \xi_1, \xi_2, \xi_3) \\ &= \int (\alpha_{jk})^{-2} \psi_1 dx + \Phi_{1,k}(t, y, z, \xi_1, \xi_2, \xi_3) \\ &= \int (\alpha_{jk})^{-2} \psi_2 dy + \Phi_{2,k}(t, x, z, \xi_1, \xi_2, \xi_3) \\ &= \int (\alpha_{jk})^{-2} \psi_3 dz + \Phi_{3,k}(t, x, y, \xi_1, \xi_2, \xi_3), \end{aligned} \quad (4.60)$$

bearing in mind that  $\alpha_{jk} = \frac{1}{\sqrt{t}}(f_{jk})^{\frac{1}{4}}$  and Eq. (4.23), while the  $\Phi$  functions may be fixed by demanding consistency with Eq. (4.5). This method leads to the following formulas for the complete evaluation of phase functions:

$$\Phi_{0,k}(x, y, z, \xi_1, \xi_2, \xi_3) = x\xi_1 + y\xi_2 + z\xi_3 - \lim_{t \rightarrow 0} \int (f_{jk})^{-\frac{1}{2}} \tilde{\psi}_0 dt, \quad (4.61)$$

$$\lim_{t \rightarrow 0} \Phi_{l,k}(t, X^l, \xi_1, \xi_2, \xi_3) = x\xi_1 + y\xi_2 + z\xi_3 - \lim_{t \rightarrow 0} \int (f_{jk})^{-\frac{1}{2}} t^{2p_l} \tilde{\psi}_l dx^l, \quad \forall l = 1, 2, 3, \quad (4.62)$$

where  $X^l$  denotes the triplet  $x^1 = x, x^2 = y, x^3 = z$  deprived of the  $l$ -th coordinate, and no summation over  $l$  is performed on the right-hand side.

## 5. Concluding remarks

The work in Ref. [6] succeeded in the difficult task of setting up a solution algorithm for defining and solving self-dual gravity field equations to first order in the non-commutativity matrix. However, precisely the first building block, i.e. the task of solving the scalar field equation in a classical self-dual background was only briefly described.

This incompleteness has been taken care of in the present paper for the case of Kasner spacetime, first with a particular choice of Kasner parameters:  $p_1 = 1, p_2 = p_3 = 0$ . The physics-oriented literature had devoted efforts to evaluating quantum propagators for a massive scalar field in the Kasner universe [16], but the relevance for the classical wave equation of the mathematical work in Refs. [7,9,10,11,23,24] had not been appreciated, to the best of our knowledge. As far as we know, our original results in Secs. III and IV are substantially new. We have indeed evaluated the bicharacteristics of Kasner spacetime in terms of elliptic integrals of first, second and third kind, while the nonlinear system for obtaining amplitude and phase functions in the scalar parametrix<sup>e</sup> has been first mapped into Eqs. (4.18)-(4.20), a set of equations that holds in any curved spacetime. Furthermore, the nonlinear equation (4.20) has been mapped into Eq. (4.31), and the latter has been solved with the help of the Adomian method, arriving at Eqs. (4.53)-(4.59).

There is however still a lot of work to do, because the proof that the asymptotic expansion of  $f_{jk} \equiv (\tilde{\alpha}_{jk})^4$  is of the Poincaré type or, instead, only involves finitely many terms, might require new insight from asymptotic and functional analysis. This adds evidence in favour of noncommutative gravity needing the whole apparatus of classical mathematical physics for a proper solution of its field equations (see also the work in Ref. [26], where Noether-symmetry methods have been used to evaluate the potential term for a wave-type operator in Bianchi I spacetime).

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<sup>e</sup>We note, incidentally, that rediscovering the versatility of parametrices might lead to important progress in canonical quantum gravity, since the work in Ref. [25] obtained diffeomorphism-invariant Poisson brackets on the space of observables, i.e. diff-invariant functionals of the metric, but this relied upon exact Green functions obeying advanced and retarded boundary conditions, whereas the parametrix is what is strictly needed in the applications, and it might prove more useful in defining and evaluating quantum commutators.

### Appendix A. Assessment of our wave equation and its solution

As we know from Sec. V [6], Eq. (1.3) is a particular case of the wave equation (3.1). The operator  $P$  in Eq. (3.1) is an example of what is called, in the mathematical literature, a Fuchsian hyperbolic operator with weight 2 with respect to  $t$ . In general, the weight is  $m - k$ , and such Fuchsian hyperbolic operators read as (hereafter  $(t, x) = (t, x^1, \dots, x^n) \in [0, T] \times \mathbf{R}^n$ )

$$P(t, x, \partial_t, \partial_x) = t^k \partial_t^m + P_1(t, x, \partial_x) t^{k-1} \partial_t^{m-1} + \dots + P_k(t, x, \partial_x) \partial_t^{m-k} + P_{k+1}(t, x, \partial_x) \partial_t^{m-k-1} + \dots + P_m(t, x, \partial_x), \quad (\text{A.1})$$

subject to 10 conditions stated in Ref. [23] which specify the relation between  $k$  and  $m$ , the form of the coefficients, hyperbolicity, quadratic form associated to the operator (see below), estimates for principal part and lower order terms of the operator. When all these 10 conditions hold, one can prove the following theorem [23]:

**Theorem A1.** For any functions  $u_0(x), \dots, u_{m-k-1}(x) \in C^\infty(\mathbf{R}^n)$  and  $f(t, x) \in C^\infty([0, T] \times \mathbf{R}^n)$ , there exists a unique solution  $u(t, x) \in C^\infty([0, T] \times \mathbf{R}^n)$  such that

$$P(t, x, \partial_t, \partial_x)u(t, x) = f(t, x) \text{ on } [0, T] \times \mathbf{R}^n, \quad (\text{A.2})$$

$$\partial_t^i u(t, x)|_{t=0} = u_i(x) \text{ for } 0 \leq i \leq m - k - 1, \quad (\text{A.3})$$

and the solution has a finite propagation speed.

For the operator in Eq. (3.1), the quadratic form of the general theory, obtained by replacing

$$\frac{\partial}{\partial x_j} \rightarrow i\xi_j$$

in all spatial derivatives of second order, reads as

$$S(t, \xi) = \sum_{j=1}^3 t^{-2p_j} (\xi_j)^2. \quad (\text{A.4})$$

According to Tahara, for Fuchsian hyperbolic operators, the quadratic form  $S(t, \xi)$  leading to Theorem A1 should be positive-definite as a function of  $\xi$  for any  $t > 0$ , with symmetric coefficients of class  $C^1$  on  $[0, T]$ , and such that

$$\max_{|\xi|=1} \left| \frac{\partial}{\partial t} \log S(t, \xi) \right| = O\left(\frac{1}{t}\right) \text{ as } t \rightarrow 0^+. \quad (\text{A.5})$$

For the operator in Eq. (3.1) one finds indeed

$$\frac{\partial}{\partial t} \log S(t, \xi) = -\frac{2 \sum_{j=1}^3 p_j t^{-2p_j} (\xi_j)^2}{\sum_{j=1}^3 t^{-2p_j} (\xi_j)^2}. \quad (\text{A.6})$$

Thus, bearing in mind that, when the  $p_j$  Kasner parameters are all nonvanishing, one of them is negative and the other two are positive, one obtains (on defining  $p \equiv \max\{p_j\}$  for all  $p_j > 0$ )

$$\left| \frac{\partial}{\partial t} \log S(t, \xi) \right| \sim \left| \frac{2p}{t} \right| \text{ as } t \rightarrow 0^+, \quad (\text{A.7})$$

and hence condition (A5) of the general theory is fulfilled. This is also the case of the operator in Eq. (1.3), for which

$$S(t, \xi) = t^{-2}(\xi_1)^2 + (\xi_2)^2 + (\xi_3)^2, \quad (\text{A.8})$$

which implies that

$$\left| \frac{\partial}{\partial t} \log S(t, \xi) \right| = \left| \frac{2}{t} \frac{t^{-2}(\xi_1)^2}{t^{-2}(\xi_1)^2 + (\xi_2)^2 + (\xi_3)^2} \right| = O\left(\frac{1}{t}\right) \text{ as } t \rightarrow 0^+. \quad (\text{A.9})$$

In other words, the hyperbolic equation studied in our paper can always rely upon the Tahara theorem on the Cauchy problem.

If instead we resort to the Garabedian technique of integration in the complex domain, strictly speaking, we need to assume analytic coefficients [9], which is not fulfilled, for example, by  $b = \frac{1}{t}$  in (2.19) if we replace  $t$  by a complex  $\tau = \tau_1 + i\tau_2$  and want to consider also the value  $\tau_1 = \tau_2 = 0$ . However, Ref. [9] describes the way out of this nontrivial technical difficulty. For this purpose, one considers first a more complicated, inhomogeneous equation

$$L[u] = f \quad (\text{A.10})$$

with analytic coefficients and analytic right-hand side, from which one can write down a direct analogue of the solution (2.20) in the form

$$u(t, x) = \lim_{\partial D \rightarrow T} \left[ \int_{\partial D} B[u, \mathcal{P}] + \int_D (Pf - uM[\mathcal{P}]) d\tau \wedge dy^1 \wedge dy^2 \wedge dy^3 \right], \quad (\text{A.11})$$

where  $\mathcal{P}(x, y)$  is called a *parametrix* (i.e. a distribution [12] that provides an approximate inverse) and is given by

$$\mathcal{P}(x, y) = \sum_{l=0}^{\nu} U_l(x, y) \sigma^{l-m}(x, y) + \sum_{l=0}^{\mu} V_l(x, y) \sigma^l(x, y) \log \sigma(x, y), \quad (\text{A.12})$$

in terms of the world function  $\sigma(x, y)$  of Appendix B. The notation  $\partial D \rightarrow T$  means that the manifold of integration  $D$  is supposed to approach the real domain in such a way that it folds around the characteristic conoid  $\sigma = 0$  without intersecting it. Equation (A11) defines a Volterra integral equation for the solution of the Cauchy problem. It follows that  $u$  varies continuously with the derivatives of the coefficients of Eq. (A10). Similarly, the second partial derivatives of  $u$  depend continuously on the derivatives of the coefficients of a high enough order. Thus, when they are *no longer analytic*, we may replace these coefficients by polynomials approximating an appropriate set of their derivatives in order to establish the validity of (A11) in the general case by passage to the limit. Note also that the integral equation (A11) has



a meaning in the real domain even where the partial differential equation (A10) is not analytic, since the construction of the parametrix  $\mathcal{P}(x, y)$  and of the world function  $\sigma(x, y)$  only requires differentiability of the coefficients of a sufficient order [9].

More precisely, for coefficients possessing partial derivatives of all orders, we introduce a polynomial approximation that includes enough of these derivatives to ensure that the solution of the corresponding approximate equation (A11) converges together with its second derivatives. The limit has therefore to be a solution of the Cauchy problem associated with the more general coefficients, and must itself satisfy the Volterra integral equation (A11).

## Appendix B. World function and fundamental solutions

In his analysis of partial differential equations, Hadamard discovered the importance of the *world function* [27,28,29], which can be defined as the square of the geodesic distance between two points with respect to the metric

$$g = \sum_{i,j=1}^n g_{ij} dx^i \otimes dx^j. \quad (\text{B.1})$$

In the analysis of second-order linear partial differential equations

$$N[u] = \left[ \sum_{i,j=1}^n a^{ij} \frac{\partial^2}{\partial x^i \partial x^j} + \sum_{i=1}^n b^i \frac{\partial}{\partial x^i} + c \right] u = 0, \quad (\text{B.2})$$

the first-order nonlinear partial differential equation (cf. Eq. (3.5)) for the world function  $\sigma(x, y)$  reads as [9]

$$\sum_{i,j=1}^n a^{ij} \frac{\partial \sigma(x, y)}{\partial x^i} \frac{\partial \sigma(x, y)}{\partial x^j} = \sum_{i,j=1}^n a^{ij} \frac{\partial \sigma(x, y)}{\partial y^i} \frac{\partial \sigma(x, y)}{\partial y^j} = 4\sigma(x, y), \quad (\text{B.3})$$

where the coefficients  $a^{ij}$  are the same as those occurring in the definition of the operator  $N$  (this is naturally the case because the wave or Laplace operator can be always defined through the metric, whose signature determines the hyperbolic or elliptic nature of the operator, as we stressed in Sec. I). The world function can be used provided that the points  $x$  and  $y$  are so close to each other that no caustics occur.

A *fundamental solution*  $S = S(x, y)$  of Eq. (B2) is a distribution [12], and can be defined to be [9] a solution of that equation in its dependence on  $x = (x^1, \dots, x^n)$  possessing, at the parameter point  $y = (y^1, \dots, y^n)$ , a singularity characterized by the representation

$$S(x, y) = \frac{U(x, y)}{(\sigma(x, y))^m} + V(x, y) \log(\sigma(x, y)) + W(x, y), \quad (\text{B.4})$$

where  $U, V, W$  are supposed to be regular functions of  $x$  in a neighbourhood of  $y$ , with  $U \neq 0$  at  $x = y$ , and where the exponent  $m$  depends on the spacetime

dimension  $n$  according to  $m = \frac{(n-2)}{2}$ . The sources of nonvanishing  $V$  are either a mass term in the operator  $N$  [30] or a nonvanishing spacetime curvature [31]. The term  $V \log(\sigma)$  plays an important role in the evaluation of the integral (2.20), as is stressed in Sec. 6.4 of Ref. [9].

In Kasner spacetime, the Hadamard Green function (B4) has been evaluated explicitly only with the special choice of parameters  $p_1 = p_2 = 0, p_3 = 1$  in Ref. [16]. In that case, direct integration of the geodesic equation (Appendix C) yields eventually an exact formula for the Hadamard-Ruse-Synge world function in the form [16]

$$\sigma = t_0^2 \left( r_\perp^2 - \tau^2 - \tau'^2 + 2\tau\tau' \cosh(r_3) \right), \quad (\text{B.5})$$

having defined

$$\tau \equiv \frac{t}{t_0}, \quad \tau' \equiv \frac{t'}{t_0}, \quad (\text{B.6})$$

$$r_\perp \equiv \frac{\sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}}{t_0}, \quad r_3 \equiv \frac{(x_3 - y_3)}{t_0}. \quad (\text{B.7})$$

Following our remarks at the end of Sec. II, we expect that the choice of Kasner parameters made in Sec. II would still lead to a formula like (B5) for the world function, but with

$$r_\perp \equiv \frac{\sqrt{(x_2 - y_2)^2 + (x_3 - y_3)^2}}{t_0}, \quad r_1 \equiv \frac{(x_1 - y_1)}{t_0}. \quad (\text{B.8})$$

However, as far as we know, the extension of these formulas to generic values of Kasner parameters is an open problem.

### Appendix C. World function of a Kasner spacetime

The calculation in Ref. [16] is so enlightening and relevant for our purposes that it deserves a brief summary. To begin, the geodesic equation in a Kasner spacetime with metric

$$g = -dt \otimes dt + \sum_{i=1}^3 t^{2p_i} dx^i \otimes dx^i \quad (\text{C.1})$$

is the following coupled system of nonlinear differential equations:

$$\frac{d^2 t}{d\lambda^2} + \sum_{i=1}^3 p_i t^{2p_i-1} \left( \frac{dx^i}{d\lambda} \right)^2 = 0, \quad (\text{C.2})$$

$$\frac{d^2 x^i}{d\lambda^2} + \frac{2p_i}{t} \frac{dx^i}{d\lambda} \frac{dt}{d\lambda} = 0, \quad (\text{C.3})$$

where  $\lambda$  is the affine parameter of the geodesic. Equation (C3) can be solved for  $Y^i \equiv \frac{dx^i}{d\lambda}$ , because it yields

$$\frac{d}{d\lambda} \log(Y^i) = \frac{d}{d\lambda} \log(t^{-2p_i}), \quad (\text{C.4})$$

which implies

$$\frac{dx^i}{d\lambda} = n_i t^{-2p_i}, \quad (\text{C.5})$$

having denoted by  $n_1, n_2, n_3$  three integration constants. The constancy along the geodesic of the (pseudo-)norm squared  $g_{\mu\nu} Y^\mu Y^\nu$ , where  $Y^\mu \equiv \frac{dx^\mu}{d\lambda}$  ( $\mu = 0, 1, 2, 3$ ), yields ( $\varepsilon$  being negative (resp. positive) for timelike (resp. spacelike) geodesics)

$$\varepsilon = - \left( \frac{dt}{d\lambda} \right)^2 + g_{ij} \frac{dx^i}{d\lambda} \frac{dx^j}{d\lambda} = - \left( \frac{dt}{d\lambda} \right)^2 + \sum_{i=1}^3 (n_i)^2 t^{-2p_i}, \quad (\text{C.6})$$

from which we obtain

$$d\lambda = \frac{dt}{\sqrt{\sum_{i=1}^3 (n_i)^2 t^{-2p_i} - \varepsilon}}. \quad (\text{C.7})$$

On the other hand, the world function is the square of the geodesic distance between the points  $P'$  and  $P$ , say, i.e.

$$\sigma = \varepsilon L^2, \quad L \equiv \int_{P'}^P d\lambda = \int_{\tau'}^\tau \frac{dt}{\sqrt{\sum_{i=1}^3 (n_i)^2 t^{-2p_i} - \varepsilon}}. \quad (\text{C.8})$$

Moreover, following Ref. [16], one defines

$$r_i \equiv (x^i - x'^i) = n_i \int_{\tau'}^\tau \frac{t^{-2p_i} dt}{\sqrt{\sum_{j=1}^3 (n_j)^2 t^{-2p_j} - \varepsilon}}. \quad (\text{C.9})$$

Upon considering the particular choice  $p_1 = p_2 = 0, p_3 = 1$ , and defining

$$N_{1,2}^\varepsilon \equiv (n_1)^2 + (n_2)^2 - \varepsilon, \quad (\text{C.10})$$

these formulae make it possible to re-express the integration constants in the form

$$n_k \equiv \frac{r_k}{\sqrt{\tau^2 N_{1,2}^\varepsilon + (n_3)^2} - \sqrt{\tau'^2 N_{1,2}^\varepsilon + (n_3)^2}} \quad \forall k = 1, 2, \quad (\text{C.11})$$

$$n_3 = \frac{\tau' e^{-r_3} \sqrt{\tau^2 N_{1,2}^\varepsilon + (n_3)^2} - \tau \sqrt{\tau'^2 N_{1,2}^\varepsilon + (n_3)^2}}{(\tau - \tau' e^{-r_3})}, \quad (\text{C.12})$$

where  $(r_1)^2 + (r_2)^2 \equiv r_\perp^2$ . We can now square up the product  $n_3(\tau - \tau' e^{-r_3})$  from (C12), finding eventually

$$(n_3)^2 = -\tau\tau' N_{1,2}^\varepsilon \cosh(r_3) + \sqrt{\tau^2 N_{1,2}^\varepsilon + (n_3)^2} \sqrt{\tau'^2 N_{1,2}^\varepsilon + (n_3)^2}. \quad (\text{C.13})$$

On the other hand, the geodesic distance in (C8) becomes in our case

$$L = \int_{\tau'}^{\tau} \frac{dt}{\sqrt{N_{1,2}^{\varepsilon} + \frac{(n_3)^2}{t^2}}} = \frac{1}{N_{1,2}^{\varepsilon}} \left[ \sqrt{\tau^2 N_{1,2}^{\varepsilon} + (n_3)^2} - \sqrt{\tau'^2 N_{1,2}^{\varepsilon} + (n_3)^2} \right], \quad (\text{C.14})$$

and if we square it up and then exploit (C13) we obtain

$$\sigma = \varepsilon \frac{(\tau^2 + \tau'^2 - 2\tau\tau' \cosh(r_3))}{N_{1,2}^{\varepsilon}}, \quad (\text{C.15})$$

because the terms involving products of square roots cancel each other. At this stage, we can re-express the squares of  $n_1$  and  $n_2$  from (C11), i.e.

$$(n_k)^2 = \frac{(r_k)^2 N_{1,2}^{\varepsilon}}{(\tau^2 + \tau'^2 - 2\tau\tau' \cosh(r_3))} \quad \forall k = 1, 2. \quad (\text{C.16})$$

By virtue of (C15) and (C16), we find eventually the result (B5), where the role played by  $t_0$  in the formulas has been made explicit.

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